Theoretical basis for application Ballistica X

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The inner working of the computational core of our application **Ballistica X** will be sketched here. Together with physical principles applying to long-range shooting.

1 Physical model

The basis of sufficiently accurate long-range shooting model was used in book [1]. This model is based on equation of motion for point-mass which is under effect of gravity and air resistance force. Also together with Coriolis and Euler force, which are caused by non-inertial reference frame of rotating and accelerating Earth. These two forces are not considered here, because they are not essential for the brief introduction.

Because of the point-mass approximation, we will not consider any rotation.

Our model is simple but it makes very good predictions. We will solve the equation numerically with a use of G_i functions which were experimentally measured. Each of these function includes various bullet geometries ($i \in \{1, 2, 5, 6, 7, 8\}$).

1.1 Equations of motion

Usage of Newton's Second Law of Motion $\boldsymbol{F} = m\boldsymbol{a}$ with gravity and air resistance leads to a system of differential equations. Its solution is a trajectory $\boldsymbol{x}(t)$ of the bullet dependent on time t

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{F_G} - \boldsymbol{F_R} \\ &= m\boldsymbol{g} - \frac{1}{2}\rho(y) \cdot S \cdot \|v\|^2 C_D \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \\ &= m\boldsymbol{g} - mc_{bc} \cdot \rho(y) \cdot G_i(\frac{\|\boldsymbol{v}\|}{v_{\text{sound}}}) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|},\end{aligned}$$

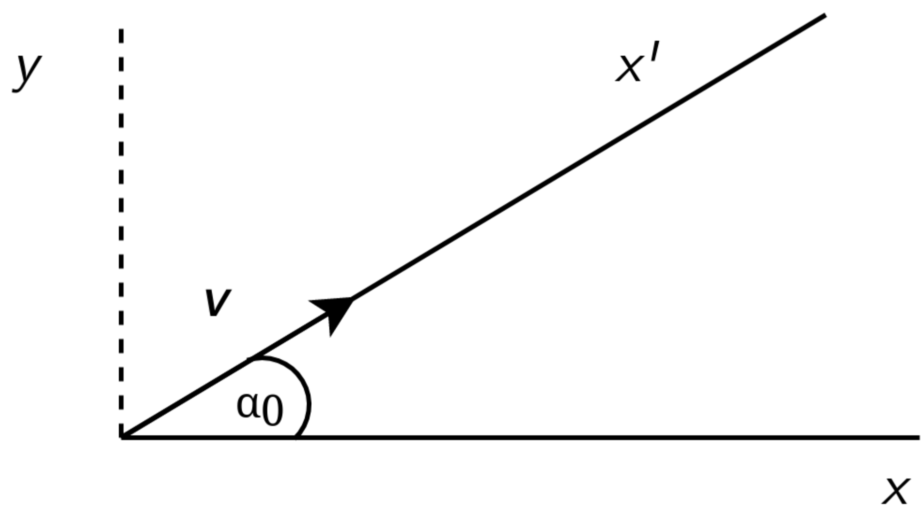
where $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{x}}$ is first (velocity), respectively second (acceleration) derivative of position $\boldsymbol{x}(t)$ with respect to time, $\boldsymbol{F_G}$ gravity and $\boldsymbol{F_R}$ air resistance force. S is the cross-section of the bullet, C_D is the bullet resistance coefficient and c_{bc} is a ballistic coefficient (in SI units), $\rho(y)$ unitless relative air density in current bullet altitude above sea level relative to air density at zero sea level, $\rho(0) = 1$. Speed of sound in air is denoted as v_{sound}.

$\boldsymbol{F_R}$ is then computed from empirically measured scalar function G_i (where i corresponds to model type - G_1 is used especially for handgun and small-caliber bullets, G_7 for longer and more aero-dynamical bullet shape, ..., it describes bullet shape). These functions depend on ratio $\frac{\|\boldsymbol{v}\|}{v_{\text{sound}}}$ - ratio of bullet velocity magnitude and speed of sound in air. We used function G_i table for velocities up to 5 MACH, available at [2].

Functions G_i are computed from these tables using linear interpolation, which is possible due to good resolution of the table.

1.2 Coordinate system

It is important to add that we chose our coordinate system to be 2-dimensional, because the problem can be examined in a plane. That is, if we neglect Coriolis and Euler force (wind effects are also neglected for now). Non-orthogonal coord. system was chosen such that x' axis is pointing in barrel direction in shooting direction. $y = y'$ axis was chosen to be normal to Earth's surface. Together they form a plane in which is embedded the bullet trajectory. We could also choose physically equal orthogonal coordinate system, which would not depend on barrel angle. But our first option will be of value in a numerical part of the computation - namely in uphill shooting ($\alpha_0 \neq 0$).



Obrázek 1: x' axis is parallel to bullet initial velocity $\boldsymbol{v}_0$. Our coordinate system will be (x', y).

2 Mathematical solution

So far, we have encumbered ourselves with physical reality. Let us transition to a solution of the obtained equations. To have a better look on a form of the equations, we will write them down using velocity $\boldsymbol{v}$ and its derivative $\dot{\boldsymbol{v}}$

$$\dot{\boldsymbol{v}} = \boldsymbol{g} - c_{bc} \cdot \rho(y) \cdot G_i\left(\frac{\|\boldsymbol{v}\|}{v_{\text{sound}}}\right)\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \tag{1}$$

which is a first order differential equation for vector function $\boldsymbol{v} = \boldsymbol{v}(t)$ (the sought velocity) in form

$$\dot{\boldsymbol{v}} = f(t, \boldsymbol{v}).$$

Because of the continuity[1] of all the function on right-hand side, there exists a solution of our equation (1). Uniqueness[2] of our solution is not generally ensured but we will not examine it and we will assume uniqueness of the solution instead[3].

2.1 Numerical solution

After assumption of existence and uniqueness of the solution, we can try to numerically find the solution of (1). Separating it into different components of velocity $\boldsymbol{v} = (v^1, v^2)$ we can see

$$\begin{aligned}
\dot{v^1} &= -c_{bc} \cdot \rho(y) \cdot G_i\left(\frac{\|\boldsymbol{v}\|}{v_{\text{sound}}}\right)\frac{v^1}{\|\boldsymbol{v}\|} \\
\dot{v^2} &= -g - c_{bc} \cdot \rho(y) \cdot G_i\left(\frac{\|\boldsymbol{v}\|}{v_{\text{sound}}}\right)\frac{v^2}{\|\boldsymbol{v}\|}.
\end{aligned}$$

This system can be directly computed but for practical reasons we would like to know all of the variables' dependencies on coordinate x or x', not with respect to time t. We can then compute reasonable table of variables for a point of the trajectory corresponding to a particular distance (see below). This can be solved by using a relation $\tilde{v}(x) = v(t(x))$ and differentiating both sides w.r.t. to x, where velocity $\tilde{v}$ is a function of distance, on the other hand v is a function of time. Then we can use a simple substitution (for ex. one from a book [1], page 55). System will be then solved using modified Euler method[4].

Let us repeat that we fire the bullet from $x' = 0$, we know the initial velocity v_0 and angle α_0; that is $v'^1(0) = v_0$, $v'^2(0) = 0$.

2.2 Shooting uphill

This way, we solved even shooting with non-zero angle α_0. Note that we wrote only v instead of v' in all equations! If we want to write all final variables as a function of

[1]Even in case when empirical function G_i is not continuous, we use linear interpolation which is surely continuous.

[2]non-existence of different solution

[3]continuous differentiability of f w.r.t. to velocity components v^1, v^2 is sufficient for example

[4]description can be found for ex. here https://www.physics.utah.edu/~detar/phys6720/handouts/ode/ode/node3.html

one variable x', we have to convert these equations to dashed coordinate system (x', y). Differentiating $x' = x \cos \alpha_0$ w.r.t time t obtains us relations for v' which we can substitute into our equations.

Everything could also be computed in original coordinate system (x, y) and in the end, we would get all variables dependent on „ground distance"[5] x. Coordinate axis were altered to compute ballistic tables with fixed x' step. This can be ensured by using this coord. system transformation.

3 Ballistic coefficient computation

In previous section, means of computing ballistic tables/parameters from know initial conditions were given (ballistic coefficient, initial velocity, ...). Computation of unknown ballistic coefficient from known trajectory (radar data for example) will be now discussed.

Assume we know the initial velocity and velocity at x metres. We can imagine that there is a signal gate at x metres which is able to precisely measure the bullet velocity. Also assume we know atmospheric conditions at the time of bullet fire. Then we can precisely compute a ballistic coefficient c_{bc} for given buller.

A solution in form of an iterative method will be considered. We will be drawing nearer to the real ballistic coefficient in every iteration. The method will stop when we achieve given precision. The method needs a relatively good initial guess of the ballistic coefficient to converge. We will choose this initial guess for G_1 BC to be

$$c^*_{bc,G_1} = \frac{0.0052834 \cdot x}{\sqrt{v_0} - \sqrt{v_x}},$$

where v_0 is the initial velocity and v_x is the measured velocity at x. Ballistic coefficients of other kinds will be approximated based on other experimental data. For a certain range of velocities, we have $c_{bc,G_7} \approx 2c_{bc,G_1}$. This guess does not have to be very precise because it is only the first point in the method. Then we will make a forward computation (compute an expected velocity at x based on this BC). If it differs from the measured velocity less then our tolerance/precision, the method stops. If not, we correct our current guess of the BC and the process is repeated. Correction for new c_{bc} can be done for ex. by using Newton's method.

The same process could be made with bullet elevation at x instead of its velocity.

Ballistic coefficient is being used in two forms - c_{bc} in SI units (suitable for computation) and C_{bc} with slightly different definition which is typically given in units $[C_{bc}] = \text{lbs/in}^2$. This coefficient C_{bc} is used on the market and is defined as

$$C_{bc} = \frac{m}{d^2 i},$$

where m is mass of the bullet (lbs), d diameter of bullet (in) and i dimensionless shape coefficient.

[5] projection of direct laser distance to a target

Conversion relation between the two is given by

$$C_{bc} = \frac{1.4222}{c_{bc}}.$$

More information about ballistic coefficient: https://en.wikipedia.org/wiki/Ballistic_coefficient#Ballistics.

Reference

[1] George Klimi. *Exterior ballistics: a new approach.* Xlibris Corp, 2010.

[2] JBM Ballistics. Drag functions, 2013. Available online at: https://www.jbmballistics.com/ballistics/downloads/downloads.shtml.